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Resonant growth of three-dimensional disturbances in plane Poiseuille flow

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A linear mechanism for growth of three-dimensional perturbations on plane Poiseuille flow is investigated. The mechanism, resonant forcing of vertical vorticity waves by Tollmien–Schlichting waves, leads to an algebraic growth for small times. Eventually, viscous damping becomes dominant and the disturbance decays. The resonance occurs only at discrete points in the wave-number space. Nine resonances have been investigated. For these, the phase velocities range from 0·67 to 0·81 of the centre-line velocity. The lowest Reynolds number for which the resonance can occur is 25. The strongest resonance appears only above a Reynolds number of 341. Also, two cases of degeneracy in the Orr–Sommerfeld dispersion relationship have been found.

1. Introduction

The linear stability properties of viscous parallel shear flows have generally been analysed in terms of the growth rates for wave-like disturbances. These growth rates are determined from the Orr–Sommerfeld equation. Motivated by Squire's theorem, only waves travelling in the direction of the mean stream are generally considered. For plane Poiseuille flow, calculations have yielded a critical Reynolds number, based on the centre-line velocity and the channel half width, for linear stability of about 5770 (Orszag 1971; Lankin, Ng & Reid 1978).

Transition to turbulence in plane Poiseuille flow has been investigated experimentally by Davies & White (1928), Narayanan & Narayana (1967), Patel & Head (1969), Karnitz, Potter & Smith (1974) and Nishioka, Iida & Ichikawa (1975). Their results are summarized in table 1. The experimental results clearly indicate that the transition Reynolds number depends strongly on the initial disturbance level. Nishioka *et al.* (1975) found that small disturbances behave as predicted by linear stability theory, that subcritical instability occurs when the disturbance level is above a certain threshold value, and that the flow can be maintained laminar up to a Reynolds number of 8000 by reducing the background turbulence level to 0·05 %.

The existence of a nonlinear subcritical instability mechanism for plane Poiseuille flow has been demonstrated through calculations incorporating weak nonlinear effects by, among others, Stewartson & Stuart (1971) and Itoh (1974, 1977, 1980). For a discussion of earlier nonlinear stability theories, the reader is referred to the review articles by Stuart (1971) and Stewartson (1975). Numerical experiments by Orszag

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Source	R_{trans}
Davies & White (1928)	1080
Karnitz, Potter & Smith (1974)	5025
Narayanan & Narayana (1967)	1425
Nishioka, Iida & Ichikawa (1975)	8000
Patel & Head (1969)	1035

TABLE 1. Transition Reynolds numbers for plane Poiseuille flow in various experiments. The Reynolds number is based on maximum velocity and half-width of channel.

& Kells (1980) and Patera & Orszag (1980) have confirmed the existence of the subcritical instability. Their direct numerical integrations of the Navier–Stokes equation suggest that finite-amplitude two-dimensional disturbances are unstable for Reynolds numbers larger than about 2800. Furthermore, finite amplitude two-dimensional neutrally stable states seemed to be explosively unstable to three-dimensional perturbations for subcritical Reynolds larger than about 1000.

The possibility of a linear resonance mechanism between the vertical vorticity eigenmodes and the Tollmien–Schlichting waves has recently been pointed out and explored for plane Couette flow by Gustavsson & Hultgren (1980). The direct resonance mechanism leads to linear growth for small times and can lead to large amplitudes before viscous damping becomes dominant. It is the purpose of the present paper to investigate this mechanism for the case of plane Poiseuille flow. As will be shown, there is a distinct difference in the occurrence of the resonances as compared to the case of plane Couette flow. Here, the resonances occur only for certain discrete values of the parameters k and αR, where k is the modulus of the wavenumber vector and αR is the product of the streamwise wavenumber and the Reynolds number. This indicates that, for a fixed Reynolds number, the direct resonance mechanism is very selective with respect to the spanwise scale.

2. Formulation

The co-ordinate system used is a Cartesian system (x, y, z). The x, y and z axes are in the streamwise, the vertical and the spanwise directions, respectively. The solid boundaries are located at $y = \pm 1$, and the dimensionless steady mean flow is given by

$$U(y) = 1 - y^2. \tag{1}$$

The development of small three-dimensional disturbances on the mean flow is governed by the following differential equations (Gustavsson & Hultgren 1980):

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} = \frac{1}{R}\nabla^4 v, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\omega_y - \frac{1}{R}\nabla^2\omega_y = -U'\frac{\partial v}{\partial z}, \tag{3}$$

where v and ω_y are the vertical velocity and vertical vorticity, respectively, of the perturbation flow field. $R = U_0 h/\nu$ is the Reynolds number, where U_0 is the centreline velocity, h is the channel half width and ν is the kinematic viscosity. The prime denotes differentiation with respect to y and ∇^2 is the Laplacian. The boundary conditions are

$$v = \frac{\partial v}{\partial y} = 0 \quad \text{at} \quad y = \pm 1, \tag{4a}$$

$$\omega_y = 0 \quad \text{at} \quad y = \pm 1. \tag{4b}$$

Equation (2) subject to the boundary conditions (4*a*) has wave-like solutions, the Tollmien–Schlichting modes. For two-dimensional perturbations, equation (2) gives a full description of the dynamics. For three-dimensional disturbances, in addition, the forced problem (3) must be solved. The homogeneous operator in (3) subject to (4*b*) also has wave-like solutions. Thus, a resonant response can occur if a Tollmien–Schlichting wave has the same wavenumber vector and eigenfrequency as a vertical vorticity mode. The formal solution of the initial-value problem, obtained by standard transform methods, is presented in Gustavsson & Hultgren (1980). The analysis in this paper follows their notation closely. For more details of the derivations, the reader is referred to that paper. Their solution shows that the resonant part of the transformed vertical vorticity, ψ_r, is given by

$$\psi_r = i\beta R \frac{\chi_1}{E} \int_{-1}^{+1} U' \phi \chi_2 dy, \tag{5}$$

where β is the spanwise wavenumber. χ_1 and χ_2 are homogeneous solutions to the transformed vertical vorticity equation. ϕ is the transform solution for the vertical velocity component and it is obtained by solving an inhomogeneous Orr–Sommerfeld equation. This latter solution has poles in the Laplace transform plane corresponding to the eigenvalues of the Orr–Sommerfeld equation. $E = 0$ is the eigenvalue relation for the vertical vorticity modes. Resonant forcing occurs if an Orr–Sommerfeld pole coincides with a pole given by $E = 0$. This double pole leads to a temporal behaviour of the form $te^{s_0 t}$, where s_0 is the pole.

The presence of the resonance phenomenon is thus investigated by studying the following eigenvalue problems:

$$\psi'' - [k^2 + i\alpha R(U - c_1)]\psi = 0, \tag{6a}$$

$$\psi(\pm 1) = 0, \tag{6b}$$

and

$$\phi^{iv} - 2k^2\phi'' + k^4\phi - i\alpha R[(U - c_2)(\phi'' - k^2\phi) - U''\phi] = 0, \tag{7a}$$

$$\phi(\pm 1) = \phi'(\pm 1) = 0. \tag{7b}$$

Here, α is the streamwise wave number and $k = (\alpha^2 + \beta^2)^{\frac{1}{2}}$. The eigenvalue problem (6) only has exponentially damped solutions (Davey & Reid 1977). Therefore, the resonant response will eventually decay. The maximum amplitude is attained at a time of the order $|\alpha c_I|^{-1}$ and is proportional to $|\alpha c_I|^{-1}$, where c_I is the imaginary part of the eigenvalue.

In the next section, the results of the numerical investigation are presented. Because the eigenvalues c_1 and c_2 depend on two parameters, k and αR, it was judged necessary to somewhat limit the study. Only temporal eigenvalues, i.e. α and k are both real, were therefore considered. The spatial problem is an interesting extension of the present investigation, but it is left for future studies. Another limitation is that only exact resonances were studied. However, for the amplitude to become large, it is sufficient that the eigenvalues are close. This near resonance was found to be a fairly common phenomenon. A further simplification is possible because the basic velocity is symmetric with respect to the centre-line. The eigenfunctions to both eigenvalue problems are then either symmetric or anti-symmetric. Since U' is anti-symmetric, χ_2 and ϕ must have opposite symmetry properties for the integral in (5) to be non-zero. Furthermore, $\chi_1 = -\chi_2$ when $E = 0$. Therefore, the resonances are characterized by the symmetry properties of χ_1 (or χ_2). For the purpose of exposition, nine resonances have been investigated. It is believed that an infinite number of resonances exists, however.

3. Numerical results

The two eigenvalue problems (6) and (7) were solved numerically by using the same technique as in Gustavsson & Hultgren (1980). The numerical integration of the differential equations started at the centre-line of the channel with initial conditions corresponding to the linearly independent solutions that had the desired symmetry properties. The eigenvalues obtained for the Orr–Sommerfeld equation were compared with the ones given by Orszag (1971), and agreement to within the required decimal places was demonstrated.

In order to gain insight about the relationship between the eigenvalues c_1 and c_2, their location in the complex c plane was determined for various αR values with k equal to unity. The results are shown in figure 1 for symmetric χ_1 and in figure 2 for anti-symmetric χ_1. The eigenvalues are labelled such that a capital letter indicates an Orr–Sommerfeld eigenvalue and 's' or 'a' means symmetric or anti-symmetric eigenmode, respectively. The modes are numbered according to increasing damping rate at $\alpha R = 100$. It is seen that, for the higher modes, c_1 and c_2 form pairs that move upwards along the line on which $c_R = \frac{2}{3}$ as αR is increased. As αR becomes sufficiently large, the eigenvalues approach either the origin or $c_R = 1$. c_R is the phase velocity. This is in agreement with the results obtained by Grosch & Salwen (1968). For the Orr–Sommerfeld eigenvalues, the $c_R = \frac{2}{3}$ line and the two branches are generally denoted S, A and P, respectively (Mack 1976). It was found in two cases that, depending on the value of k, the eigenvalue c_2 could approach either branch as αR was increased. This indicates the presence of a degeneracy in the dispersion relationship. For the vertical vorticity eigenvalues, the behaviour is simpler in that consecutive eigenvalues along the S branch alternate among the A and P branches as αR increases.

The search for resonances started with the mode pair (s2, A 1), but it was found that these eigenvalues always differ as αR is varied. The next pair of eigenvalues considered was (s3, A 2). As αR increases, both eigenvalues belong to the P branch and were found to be close to each other. Curves in the k–αR plane on which the real and the imaginary parts, respectively, of the two eigenvalues are equal are shown in figure 3. The two curves are seen to cross at $k = 5{\cdot}7$ and $\alpha R = 147$. More accurate calculations give the following values for the resonance:

$$k = 5{\cdot}7942 \quad \text{and} \quad \alpha R = 147{\cdot}07 \quad \text{with} \quad c = 0{\cdot}69256 - 0{\cdot}57474i.$$

A second resonance for the same mode pair was found at

$$k = 1{\cdot}0153 \quad \text{and} \quad \alpha R = 345{\cdot}77 \quad \text{with} \quad c = 0{\cdot}80942 - 0{\cdot}19269i.$$

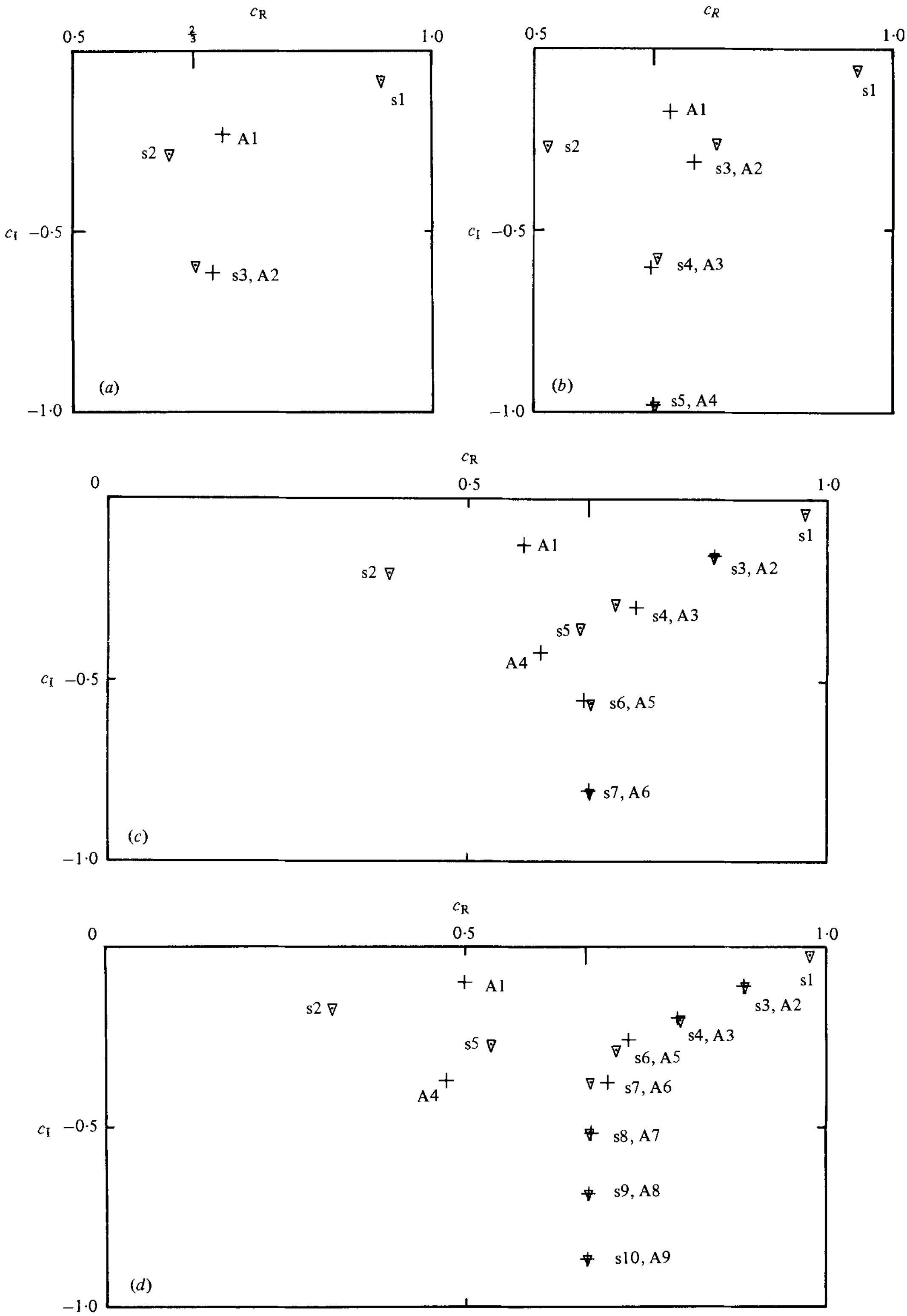
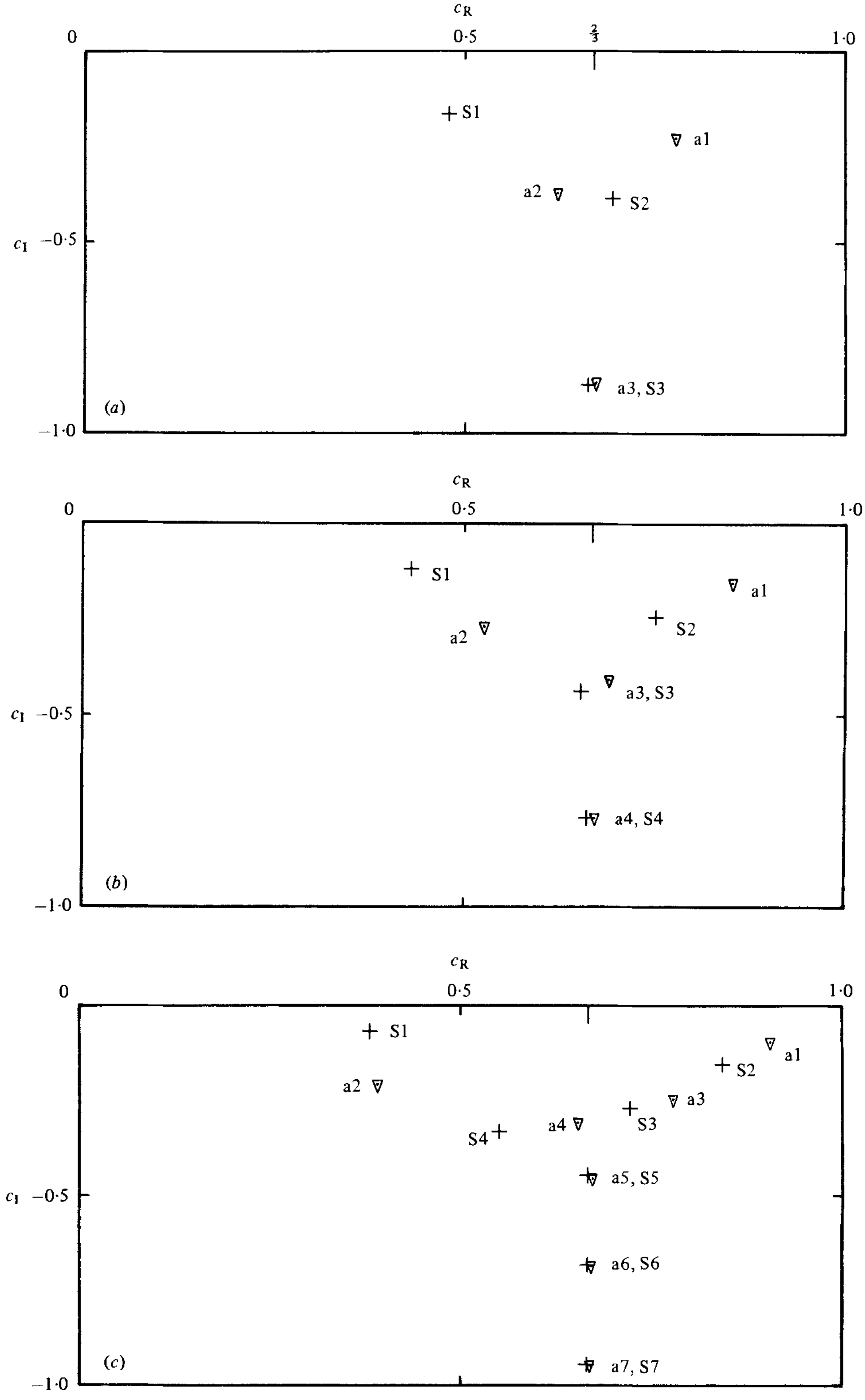


FIGURE 1. Location in the complex c plane of c_1 (∇) and c_2 (+) for symmetric χ_1 and antisymmetric ϕ. $k = 1$. (a) $\alpha R = 100$, (b) $\alpha R = 200$, (c) $\alpha R = 500$, (d) $\alpha R = 1000$.



Caption for Figures 2(*a*), (*b*) and (*c*) on opposite page.

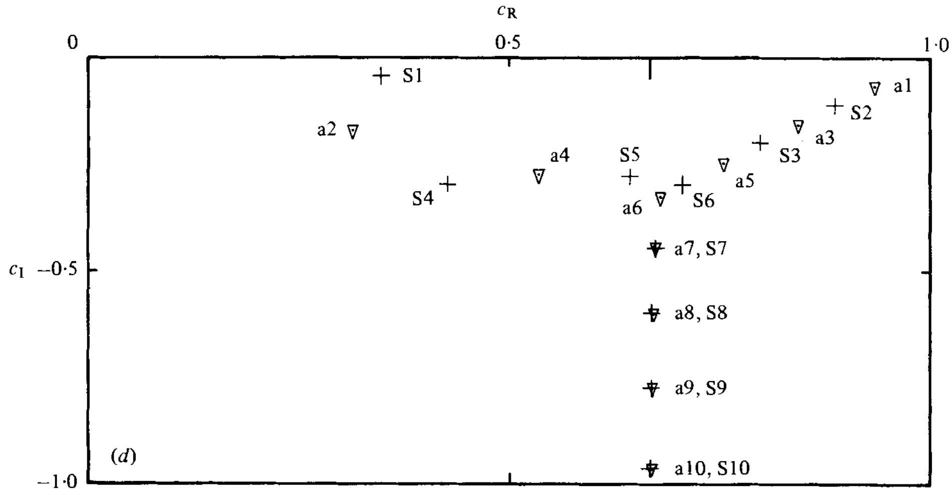


FIGURE 2. Location in the complex c plane of c_1 (∇) and c_2 (+) for anti-symmetric χ_1 and symmetric ϕ. $k = 1$. (*a*) $\alpha R = 100$, (*b*) $\alpha R = 200$, (*c*) $\alpha R = 500$, (*d*) $\alpha R = 1000$.

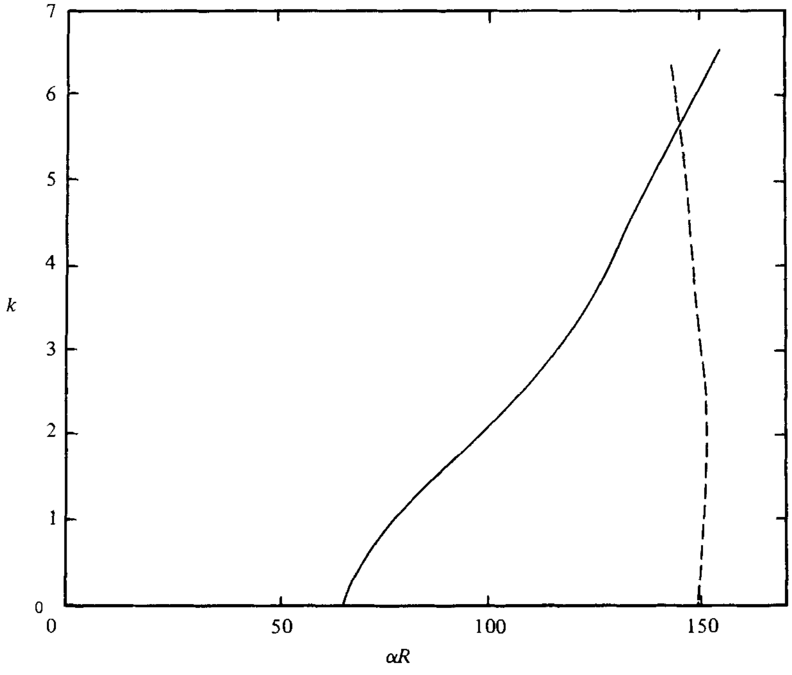
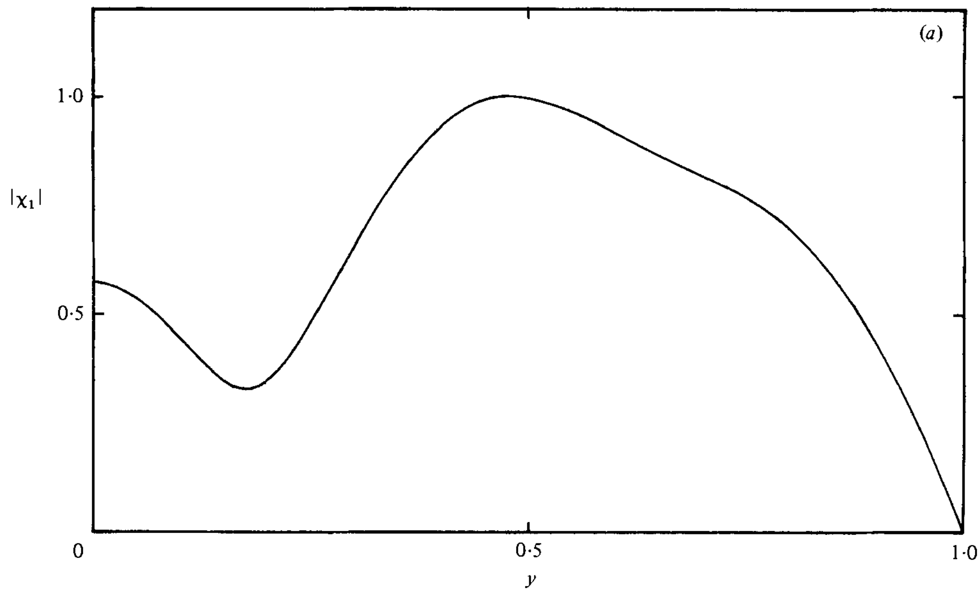


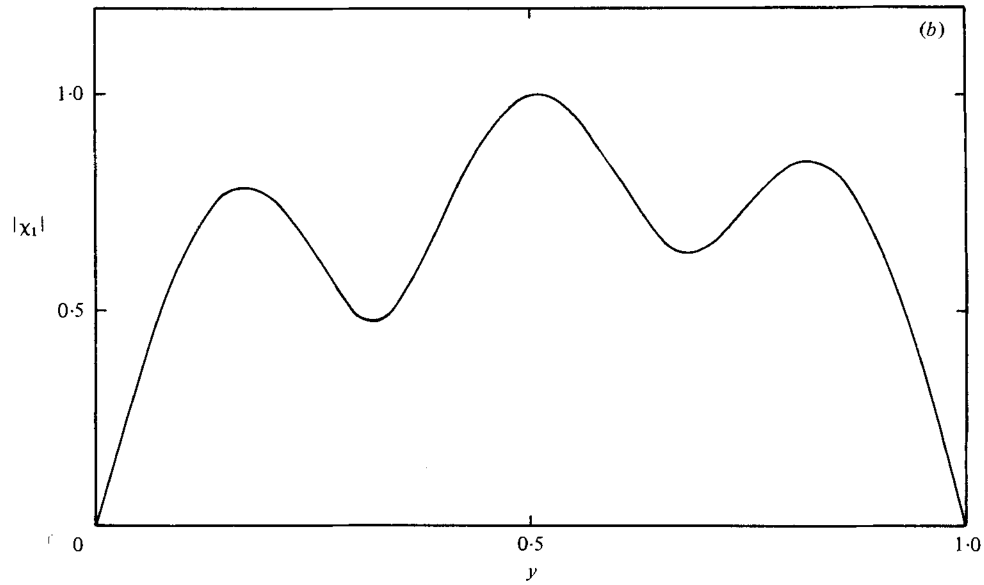
FIGURE 3. The dashed and solid lines represent curves on which the real and the imaginary parts, respectively, are equal for the eigenvalues corresponding to the modes s3 and A2.

Resonance was also found to occur for the mode pairs (s4, A 3) and (s8, A 7). Resonance was not found for the mode pairs (s5, A 4), (s6, A 5) and (s7, A 6) even though the eigenvalues can be very close. For example, for $k = 1{\cdot}5$ and $\alpha R = 250$, the absolute value of the difference between the eigenvalues s5 and A 4 is $4{\cdot}5 \times 10^{-3}$. This may be considered as effectively a resonance. In addition, it was observed that the Orr–Sommerfeld eigenvalues A 4 and A 5 seem to coincide for $k \simeq 2{\cdot}8$ and $\alpha R \simeq 620$. If

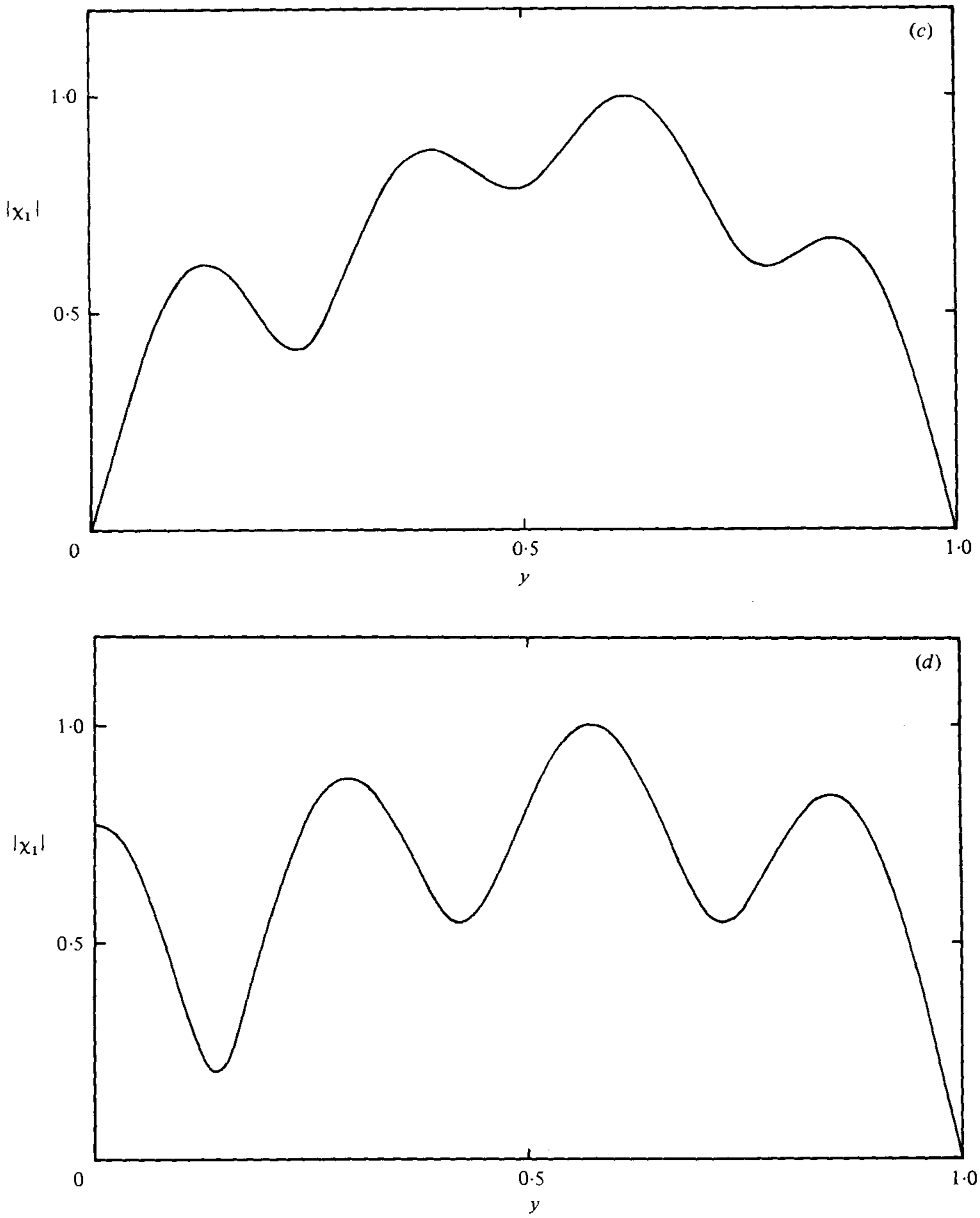
Type	Mode pair	R_{cr}	k	αR	c	$\|\alpha c_I\|^{-1}$ ($R = 1000$)
S	(s3, A2)	25·04	5·7942	145·07	0·69256–0·57474i	11·99
A	(a3, S3)	78·56	1·4777	116·09	0·67182–0·75854i	11·36
A	(a4, S4)	95·25	2·3432	223·19	0·67148–0·70405i	6·36
S	(s4, A3)	151·75	0·8166	123·92	0·67090–0·96107i	8·40
S	(s8, A7)	205·32	6·8111	1398·48	0·68073–0·37506i	1·91
S	(s3, A2)	340·55	1·0153	345·76	0·80942–0·19268i	15·01
A	(a8, S8)	591·89	1·8663	1104·65	0·67195–0·53934i	1·68
A	(a7, S7)	895·14	1·0525	942·14	0·67277–0·47551i	2·23
S	(s8, A7)	1417·11	0·6312	894·48	0·67137–0·58806i	not active

TABLE 2. Characteristics of resonances in plane Poiseuille flow. Type indicates the symmetry of χ_1 ('A', anti-symmetric; 'S', symmetric).





Caption for Figures 4(a) and (b) on p. 262.



Caption for Figures 4(*c*) and (*d*) on p. 262.

the eigenmodes corresponding to these eigenvalues are not orthogonal, this degeneracy leads to a resonant growth of the vertical velocity alone.

For anti-symmetric χ_1, the investigation showed that the mode pairs (a3, S3), (a4, S4), (a7, S7) and (a8, S8) can exhibit resonance, whereas the mode pairs (a5, S5) and (a6, S6) do not. It was also found that the Orr–Sommerfeld modes S5 and S6 seem to coincide somewhere in the region $1 < k < 2$ and $550 < \alpha R < 600$.

Table 2 summarizes the characteristics of the resonances found. In order to illustrate the achievable amplitude amplifications, the quantity $|\alpha c_1|^{-1}$ has also been

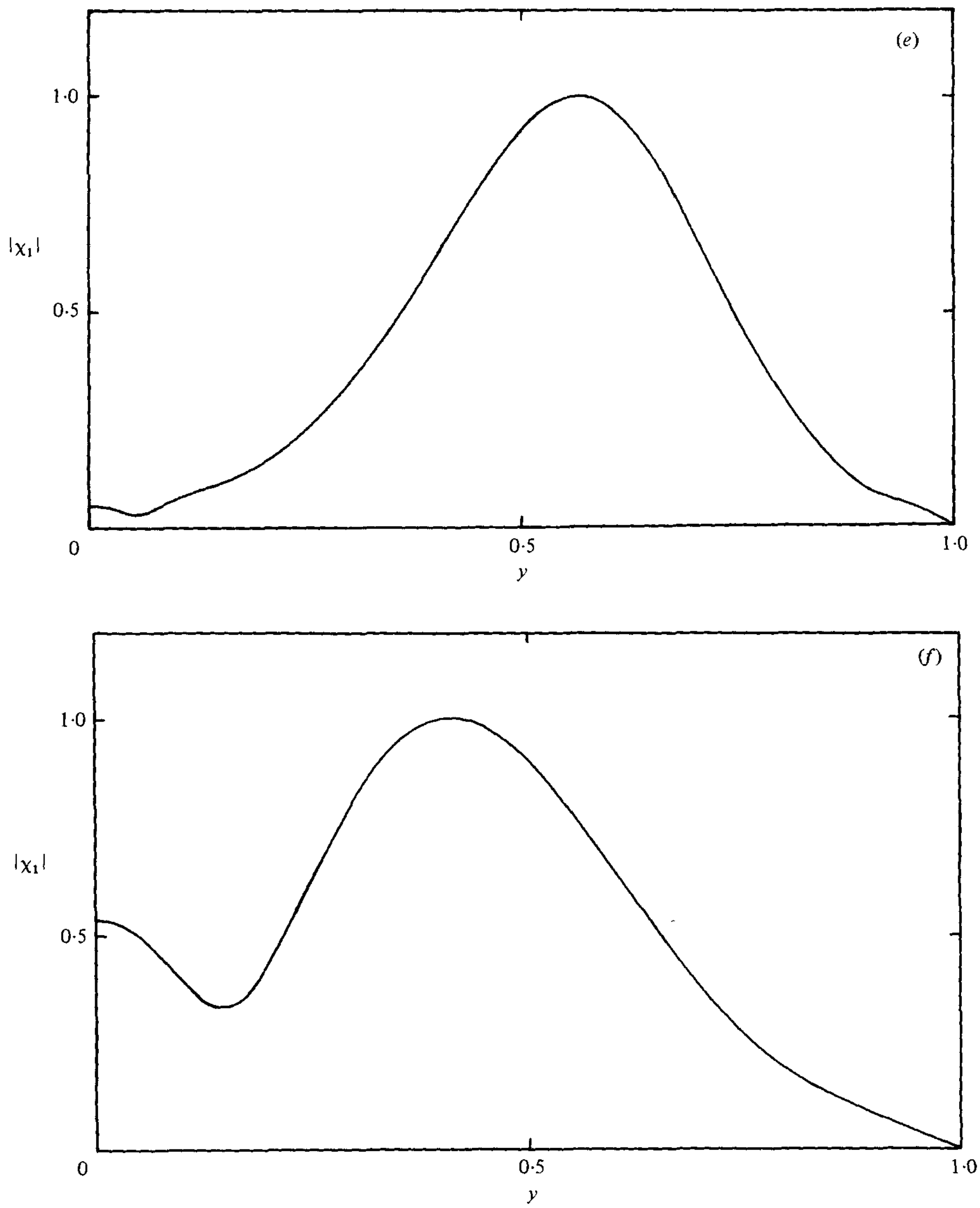


FIGURE 4. Amplitude distribution of the eigenfunction for the six first resonances in table 2. (a) (s3, A2), (b) (a3, S3), (c) (a4, S4), (d) (s4, A3), (e) (s8, A7), (f) (s3, A2).

tabulated for $R = 1000$. Finally, the amplitude distribution of χ_1 for six of the resonances are shown in figure 4.

4. Discussion

The resonances occur only at discrete points in the k–αR plane. This is in contrast to the case for plane Couette flow (Gustavsson & Hultgren 1980) where the resonances found so far occur along curves in the k–αR plane. Because $\alpha \leqslant k$, it follows that for each resonance there is a critical Reynolds number under which it cannot occur. This critical Reynolds number is defined by

$$R_{cr} = \frac{(\alpha R)_r}{k_r}. \tag{8}$$

The subscript denotes the value of the parameter at resonance. Table 2 also gives the critical Reynolds numbers for the resonances. There is no upper limit on the Reynolds number. It can be increased as long as the product αR is kept constant. Thus, as R increases, α decreases and the spanwise wave number, β, tends to k_r. Thus, at high Reynolds numbers, structures elongated in the streamwise direction and with distinct spanwise scales are likely formed. Also, the growth becomes linear for all times as $R \to \infty$.

The resonances that have been presented are exact, i.e. the eigenvalues c_1 and c_2 can be made to coincide to any required degree of accuracy. However, for a resonant-like behaviour to occur, it is sufficient that the eigenvalues are close. This indicates that even those mode pairs that do not exhibit resonance may produce large amplitudes at some wavenumber combination(s).

A simple calculation shows that, for given fluid and geometry, the dimensional resonance frequency is independent of the flow speed. From table 2, it is observed that the ratio $\alpha c_{\mathrm{R}}/|\alpha c_{\mathrm{I}}|$ is of order unity for most of the resonances. Only parts of a full oscillation will therefore be completed before the amplitude maximum is reached. This fact may complicate efforts to detect the resonances experimentally.

The critical Reynolds numbers for the found resonances are all small compared to the lowest Reynolds number ($\sim$ 1000) for which transition has been reported to occur for plane Poiseuille flow. At this Reynolds number, eight resonances are active. This suggests that nonlinear interactions between different resonances and/or secondary instabilities triggered by the resonances could play a role in the transition process.

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REFERENCES

DAVEY, A. & REID, W. H. 1977 On the stability of stratified viscous plane Couette flow. Part 1. Constant buoyancy frequency. *J. Fluid Mech.* **80**, 509.

DAVIES, S. J. & WHITE, C. M. 1928 An experimental study of the flow of water in pipes of rectangular section. *Proc. Roy. Soc.* A **119**, 92.

GROSCH, C. E. & SALWEN, H. 1968 The stability of steady and time-dependent plane Poiseuille flow. *J. Fluid Mech.* **34**, 177.

GUSTAVSSON, L. H. & HULTGREN, L. S. 1980 A resonance mechanism in plane Couette flow. *J. Fluid Mech.* **98**, 149.

ITOH, N. 1974 Spatial growth of finite wave disturbances in parallel and nearly parallel flows. Part 1. The theoretical analysis and numerical results for plane Poiseuille flow. *Trans. Japan Soc. Aero. Space Sci.* **17**, 160.

ITOH, N. 1977 Nonlinear stability of parallel flows with subcritical Reynolds numbers. Part 1. An asymptotic theory valid for small amplitude disturbances. *J. Fluid Mech.* **82**, 455.

ITOH, N. 1980 Three-dimensional growth of finite wave disturbances in plane Poiseuille flow. *Trans. Japan Soc. Aero. Space Sci.* **23**, 91.

KARNITZ, M. A., POTTER, M. C. & SMITH, M. C. 1974 An experimental investigation of a plane Poiseuille flow. *Trans. A.S.M.E.I., J. Fluids Engng* **96**, 384.

LANKIN, W. D., NG, B. S. & REID, W. H. 1978 Approximations to the eigenvalue relation for the Orr–Sommerfeld problem. *Phil. Trans. Roy. Soc.* A **289**, 347.

MACK, L. M. 1976 A numerical study of the temporal eigenvalue spectrum of the Blasius boundary layer. *J. Fluid Mech.* **73**, 497.

NARAYANAN, M. A. B. & NARAYANA, T. 1967 Some studies on transition from laminar to turbulent flow in a two-dimensional channel. *Z. angew. Math. Phys.* **18**, 642.

NISHIOKA, M., IIDA, S. & ICHIKAWA, Y. 1975 An experimental investigation of the stability of plane Poiseuille flow. *J. Fluid Mech.* **72**, 731.

ORSZAG, S. A. 1971 Accurate solution of the Orr–Sommerfeld stability equation. *J. Fluid Mech.* **50**, 689.

ORSZAG, S. A. & KELLS, L. S. 1980 Transition to turbulence in plane Poiseuille and plane Couette flow. *J. Fluid Mech.* **96**, 159.

PATEL, V. C. & HEAD, M. R. 1969 Some observations on skin friction and velocity profiles in fully developed pipe and channel flows. *J. Fluid Mech.* **38**, 181.

PATERA, A. T. & ORSZAG, S. A. 1980 Subcritical transition to turbulence in plane channel flows. *Symp. Transition and Turbulence, University of Wisconsin, Madison.*

STEWARTSON, K. 1975 Some aspects of nonlinear stability theory. *Fluid Dyn. Trans. Polish Acad. Sci.* **7**, 101.

STEWARTSON, K. & STUART, J. T. 1971 A non-linear instability theory for a wave system in plane Poiseuille flow. *J. Fluid Mech.* **48**, 529.

STUART, J. T. 1971 Nonlinear stability theory. *Ann. Rev. Fluid Mech.* **3**, 347.